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Periodic Cellular Automata of Period-2

Ganikhodjaev, N.*, Saburov, M., and Mohamad Asimoni, N. R.

Department of Computational & Theoretical Sciences,
Kulliyyah of Science,
International Islamic University Malaysia

*E-mail: nasirgani@hotmail.com, msaburov@gmail.com**

ABSTRACT

In this paper, we introduce a notion of periodic one-dimensional cellular automata of period-2 and study different properties of such kind of cellular automata. Moreover, we shall introduce an $A-$ cellular automata which is not shift commuting of any power. We calculate entropy of some periodic cellular automata with period two and present simulation for some fractal type periodic cellular automata.

Keywords: Periodic cellular automata of period-2, $A-$cellular automata, Entropy, Fractal.

1. Introduction

Cellular automata are a new approach that has been used rapidly in many areas of science. Their dynamical behavior is similar to many real systems. It has been first introduced and studied by Stainslaw Ulam and John von Neumann in 1940 for modeling biological systems. After that, cellular automata have been popularized by John H. Coway with his well-known project entitled Game of Life. Wolfram (1984) contributed in this model by showing many applications using cellular automata.

Cellular automaton is a model of a system which consist a grid of cells with each cell has a state and neighborhood. Each cell generates in time depending on their neighborhood and the set of rules. The simplest type of cellular automaton is one-dimensional automaton which used the simplest state (0 or 1) and the simplest neighborhood where the cell itself, its right and left neighborhood. This is called elementary cellular automata (Wolfram, 1983). Since there are $2^3 = 8$ possible binary states on three neighboring cell, there are $2^8 = 256$ known as the rule for the particular automaton. The rule corresponds to the one bit number on new state that affected by the three bit number on previous state.

In this paper, we introduce a new class of cellular automata, so-called periodic and $A-$cellular automata, and study their fractal structures. In addition, we calculate entropy of some period-2 cellular automata.

2. Periodic Cellular Automata of Period-2

Let $A = \{0, 1, 2, ..., r-1\}$ be an alphabet. Let

$$x = \cdots x_{-k}x_{-k+1}\cdots x_{-1} \bullet x_0x_1\cdots x_k\cdots$$

a bi-sequence over A. A full shift space $A^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in A, \forall i \in \mathbb{Z}\}$ is a set of all bi-sequences. Let $x_{[-k,k]} = x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k$ be a central block and $B_{2k+1}(A) = \{b = b_{-k}b_{-k+1}...b_{-1}b_0b_1...b_{k-1}b_k : b_i \in A\}$ be a set of all central blocks of the length $2k+1$. The shift space $A^{\mathbb{Z}}$ is a Cantor space with respect to the following metric

$$d(x,y) = \begin{cases} 2 & x_0 \neq y_0 \\ 2^{-k} & x_{[-k,k]} = y_{[-k,k]} \\ 0 & x = y \end{cases}$$

Let $\sigma : A^z \to A^z$ be a shift mapping $[\sigma(x)]_i = x_{i+1}$ for any $i \in \mathbb{Z}$.

Definition 2.1 (Hedlund (1969),Lind and Marcus (1995))**.** *A mapping* F : $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is called cellular automata if one has that* $F \circ \sigma = \sigma \circ F$*, i.e.,* $F(\sigma(x)) = \sigma(F(x))$ *for any* $x \in A^{\mathbb{Z}}$*.*

Various properties of one dimensional cellular automata including topological entropy and fractal structures were studied in a series of papers (see, for instance, Wolfram (1984), Hedlund (1969)).

In this paper, we are aiming to study periodic cellular automata of period-2.

Definition 2.2. *A mapping $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is called periodic cellular automata of period-2 if one has that $F \circ \sigma^{(2)} = \sigma^{(2)} \circ F$, i.e., $F(\sigma(\sigma(x))) = \sigma(\sigma(F(x)))$ for any $x \in A^{\mathbb{Z}}$.*

Some research has proven that cellular automata can be used as a model of real phenomena. Jafelice and Nunes (2011) discussed on the cellular automaton for a predator-prey model while Dupuis and Chopard (2003) claimed that traffic flow of cars in the city of Geneva can be simulated by cellular automata. Since the periodic cellular automata of period-2 generalises from the actual definition, we can extend these applications into more interesting cases.

Let us define the following mapping $F_{eo} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by local rules $f_e, f_o : B_{2k+1}(A) \to A$

$$[F_{eo}(x)]_i = \begin{cases} f_e(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & \text{i-even} \\ f_o(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & \text{i-odd} \end{cases} \tag{1}$$

Proposition 2.1. *A mapping $F_{eo} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by* (1) *is a continuous periodic cellular automata of period-2.*

Proof. We first show that $F_{eo} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is continuous. Let $x \in A^{\mathbb{Z}}$ and $F_{eo}(x) = y \in A^{\mathbb{Z}}$. Let $\epsilon > 0$ and $m_0 \in N$ such that $\frac{1}{2^{m_0}} < \varepsilon$. We choose $\delta > 0$ such that $\delta < \frac{1}{2^{1+k+m_0}}$ and let $u \in A^{\mathbb{Z}}$ be any element such that $d(u, x) < \delta$. This means that $x_{[-k-m_0,k+m_0]} = u_{[-k-m_0,k+m_0]}$. Let $F_{eo}(u) = v \in A^{\mathbb{Z}}$. We then get that $[F_{eo}(x)]_{[-m_0,m_0]} = [F_{eo}(u)]_{[-m_0,m_0]}$, i.e., $d(F_{eo}(u), F_{eo}(x)) < \epsilon$. This means that $F_{eo} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is continuous.

Let us now show that $F_{eo}(\sigma(\sigma(x))) = \sigma(\sigma(F_{eo}(x)))$ for any $x \in A^{\mathbb{Z}}$. In fact, we have that

$$[F_{eo}(\sigma(\sigma(x)))]_i = \begin{cases} f_e([\sigma^{(2)}(x)]_{-k+i}...[\sigma^{(2)}(x)]_i...[\sigma^{(2)}(x)]_{k+i}) & \text{i-even} \\ f_o([\sigma^{(2)}(x)]_{-k+i}...[\sigma^{(2)}(x)]_i...[\sigma^{(2)}(x)]_{k+i}) & \text{i-odd} \end{cases}$$

$$= \begin{cases} f_e(x_{-k+i+2}...x_{i+2}...x_{k+i+2}) & \text{i-even} \\ f_o(x_{-k+i+2}...x_{i+2}...x_{k+i+2}) & \text{i-odd} \end{cases}$$

$$[\sigma^{(2)}(F_{eo}(x))]_i = [F_{eo}(x)]_{i+2} = \begin{cases} f_e(x_{-k+i+2}...x_{i+2}...x_{k+i+2}) & \text{i-even} \\ f_o(x_{-k+i+2}...x_{i+2}...x_{k+i+2}) & \text{i-odd} \end{cases}$$

Therefore, $F_{eo} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a continuous periodic cellular automata of period-2. □

3. $A-$cellular Automata

In this section, we discuss a cellular automata which does not commute with the shift mapping of any power.

Let $A = \{0, 1, 2, ..., r-1\}$ be an alphabet and $f_i : B_{2k+1}(A) \to A$ be local rules for any $i \in A$.

Definition 3.1. *A mapping $F_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is called an $A-$cellular automata if it is defined as follows*

$$[F_A(x)]_i = \begin{cases} f_0(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & x_i = 0 \\ f_1(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & x_i = 1 \\ \vdots & \vdots \\ f_{r-1}(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & x_i = r-1 \end{cases} \quad (2)$$

Proposition 3.1. *A mapping $F_{eo} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by (2) is continuous.*

Proof. We show that $F_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by (2) is continuous. Let $x \in A^{\mathbb{Z}}$ and $F_A(x) = y \in A^{\mathbb{Z}}$. Let $\varepsilon > 0$ and $m_0 \in N$ such that $\frac{1}{2^{m_0}} < \epsilon$. We choose $\delta > 0$ such that $\delta < \frac{1}{2^{1+k+m_0}}$ and let $u \in A^{\mathbb{Z}}$ be any element such that $d(u, x) < \delta$. This means that $x_{[-k-m_0,k+m_0]} = u_{[-k-m_0,k+m_0]}$. Let $F_A(u) = v \in A^{\mathbb{Z}}$. We then get that $[F_A(x)]_{[-m_0,m_0]} = [F_A(u)]_{[-m_0,m_0]}$, i.e., $d(F_A(u), F_A(x)) < \epsilon$. This means that $F_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is continuous. □

Remark 3.1. *It is worth of mentioning that, in general, the $A-$cellular automata $F_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ does not commute with shift mapping of any order.*

Let $A = \{0, 1\}$ and $f_0, f_1 : B_{2k+1}(A) \to A$ be local rules.

Definition 3.2. *A mapping $F_{(0-1)} : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ defined as follows*

$$[F_{0-1}(x)]_i = \begin{cases} f_0(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & x_i = 0 \\ f_1(x_{-k}x_{-k+1}...x_{-1}x_0x_1...x_{k-1}x_k) & x_i = 1 \end{cases} \quad (3)$$

is said to be $(0-1)-$cellular automata.

4. Entropy of Periodic Cellular Automata

An entropy of periodic cellular automata $F : A^z \to A^z$ is defined by the following formula

$$h(F) = \lim_{m\to\infty} \lim_{t\to\infty} \frac{\log R(w,t)}{t} \quad (4)$$

where $R(w,t)$ is the number of distinct rectangles of width w and height t occurring in the space-time diagram (for more details see refs. Cattaneo et al. (2000), Hurd et al. (1992), Blancard et al. (1997)). In this section, we shall provide few examples for periodic cellular automata of period-2 in which their entropy might or might not be zeros.

Let $A_3 = \{-1, 0, 1\}$ and $B_3(A_3) = \{b = b_{-1}b_0b_1 : b_i \in A_3\}$ be a set of all 3-blocks. Let $f_k : B_3(A_3) \to A_3$ be a local rule $f_k(b_{-1}b_0b_1) = b_k$ for $k \in A_3$.

Let us define the following periodic cellular automatas

$$[F_{01}(x)]_i = \begin{cases} f_0(x_{i-1}x_ix_{i+1}) & \text{i=even} \\ f_1(x_{i-1}x_ix_{i+1}) & \text{i=odd} \end{cases} = \begin{cases} x_i & \text{i-even} \\ x_{i+1} & \text{i-odd} \end{cases} \quad (5)$$

$$[F_{10}(x)]_i = \begin{cases} f_1(x_{i-1}x_ix_{i+1}) & \text{i=even} \\ f_0(x_{i-1}x_ix_{i+1}) & \text{i=odd} \end{cases} = \begin{cases} x_{i+1} & \text{i-even} \\ x_i & \text{i-odd} \end{cases} \quad (6)$$

$$[F_{(-1)1}(x)]_i = \begin{cases} f_{-1}(x_{i-1}x_ix_{i+1}) & \text{i=even} \\ f_1(x_{i-1}x_ix_{i+1}) & \text{i=odd} \end{cases} = \begin{cases} x_{i-1} & \text{i-even} \\ x_{i+1} & \text{i-odd} \end{cases} \quad (7)$$

$$[F_{1(-1)}(x)]_i = \begin{cases} f_1(x_{i-1}x_ix_{i+1}) & \text{i=even} \\ f_{-1}(x_{i-1}x_ix_{i+1}) & \text{i=odd} \end{cases} = \begin{cases} x_{i+1} & \text{i-even} \\ x_{i-1} & \text{i-odd} \end{cases} \quad (8)$$

$$[F_{(-1)0}(x)]_i = \begin{cases} f_{-1}(x_{i-1}x_ix_{i+1}) & \text{i=even} \\ f_0(x_{i-1}x_ix_{i+1}) & \text{i=odd} \end{cases} = \begin{cases} x_{i-1} & \text{i-even} \\ x_i & \text{i-odd} \end{cases} \quad (9)$$

$$[F_{0(-1)}(x)]_i = \begin{cases} f_0(x_{i-1}x_ix_{i+1}) & \text{i=even} \\ f_{-1}(x_{i-1}x_ix_{i+1}) & \text{i=odd} \end{cases} = \begin{cases} x_i & \text{i-even} \\ x_{i-1} & \text{i-odd} \end{cases} \quad (10)$$

Theorem 4.1. *The entropies of periodic cellular automatas given by* (5)- (10) *are zero.*

Proof. In order to calculate the entropy, we have to calculate the number of distinct rectangles $R(w,t)$ of width w and height t occurring in space-time diagram. Let us show how to calculate $R(w,t)$ for a few periodic cellular automata

$$\left.\begin{array}{rl}
x = \cdots & x_{-2k-1}\overbrace{\boxed{x_{-2k}}\cdots\boxed{x_{-1}}\bullet\boxed{x_0}\boxed{x_1}\cdots\boxed{x_{2n-1}}\boxed{x_{2n}}}^{w}x_{2n+1}\cdots \\
F_{01}(x) = & \cdots x_{-2k}\boxed{x_{-2k}}\cdots\boxed{x_0}\bullet\boxed{x_0}\boxed{x_2}\cdots\boxed{x_{2n}}\boxed{x_{2n}}x_{2n+2}\cdots \\
F_{01}^{(2)}(x) = & \cdots x_{-2k}\boxed{x_{-2k}}\cdots\boxed{x_0}\bullet\boxed{x_0}\boxed{x_2}\cdots\boxed{x_{2n}}\boxed{x_{2n}}x_{2n+2}\cdots \\
 & \vdots \quad \vdots \quad \vdots \\
F_{01}^{(t-1)}(x) = & \cdots x_{-2k}\boxed{x_{-2k}}\cdots\boxed{x_0}\bullet\boxed{x_0}\boxed{x_2}\cdots\boxed{x_{2n}}\boxed{x_{2n}}x_{2n+2}\cdots
\end{array}\right\} t$$

$$\vdots \quad \vdots \quad \vdots$$

$$\left.\begin{array}{rl}
x = \cdots & x_{-2k-1}\overbrace{\boxed{x_{-2k}}\cdots\boxed{x_{-1}}\bullet\boxed{x_0}\boxed{x_1}\cdots\boxed{x_{2n-1}}\boxed{x_{2n}}}^{w}x_{2n+1}\cdots \\
F_{(-1)1}(x) = & \cdots x_{-2k}\boxed{x_{-2k-1}}\cdots\boxed{x_0}\bullet\boxed{x_{-1}}\boxed{x_2}\cdots\boxed{x_{2n}}\boxed{x_{2n-1}}x_{2n+2}\cdots \\
F_{(-1)1}^{(2)}(x) = & \cdots x_{-2k-1}\boxed{x_{-2k}}\cdots\boxed{x_{-1}}\bullet\boxed{x_0}\boxed{x_1}\cdots\boxed{x_{2n-1}}\boxed{x_{2n}}x_{2n+1}\cdots \\
 & \vdots \quad \vdots \quad \vdots \\
F_{(-1)1}^{(t-1)}(x) = & \cdots x_{-2k}\boxed{x_{-2k-1}}\cdots\boxed{x_0}\bullet\boxed{x_{-1}}\boxed{x_2}\cdots\boxed{x_{2n}}\boxed{x_{2n-1}}x_{2n+2}\cdots
\end{array}\right\} t$$

$$\vdots \quad \vdots \quad \vdots$$

In all cases, $R(w,t)$ is free of t and may take a value in $\{3^w, 3^{w+1}, 3^{w+2}\}$. Therefore, the entropies of periodic cellular automatas given by (5)- (10) are zero. □

Let us give an example for period cellular automata in which the entropy is not zero.

Let $A_5 = \{-2,-1,0,1,2\}$ and $B_5(A_5) = \{b = b_{-2}b_{-1}b_0b_1b_2 : b_i \in A_5\}$ be a set of all 5-blocks. Let $f_k : B_5(A_5) \to A_5$ be a local rule $f_k(b_{-2}b_{-1}b_0b_1b_2) = b_k$ for $k \in A_5$.

Let us define the following periodic cellular automatas

$$[F_{02}(x)]_i = \begin{cases} f_0(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=even} \\ f_2(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=odd} \end{cases} = \begin{cases} x_i & \text{i-even} \\ x_{i+2} & \text{i-odd} \end{cases} \quad (11)$$

$$[F_{20}(x)]_i = \begin{cases} f_2(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=even} \\ f_0(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=odd} \end{cases} = \begin{cases} x_{i+2} & \text{i-even} \\ x_i & \text{i-odd} \end{cases} \quad (12)$$

$$[F_{12}(x)]_i = \begin{cases} f_1(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=even} \\ f_2(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=odd} \end{cases} = \begin{cases} x_{i+1} & \text{i-even} \\ x_{i+2} & \text{i-odd} \end{cases} \quad (13)$$

$$[F_{21}(x)]_i = \begin{cases} f_2(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=even} \\ f_1(x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2}) & \text{i=odd} \end{cases} = \begin{cases} x_{i+2} & \text{i-even} \\ x_{i+1} & \text{i-odd} \end{cases} \tag{14}$$

Theorem 4.2. *The entropies of periodic cellular automatas given by* (11)- (14) *are* $\log 5$.

Proof. In order to calculate the entropy, we have to calculate the number of distinct rectangles $R(w,t)$ of width w and height t occurring in space-time diagram. Let us show how to calculate $R(w,t)$ for a few periodic cellular automata

$$\left.\begin{array}{rl}
x = & \cdots x_{-2k-1}\overbrace{\boxed{x_{-2k}}\cdots\boxed{x_{-1}}\bullet\boxed{x_0}\boxed{x_1}\cdots\boxed{x_{2n-1}}\boxed{x_{2n}}}^{w}x_{2n+1}\cdots \\
F_{02}(x) = & \cdots x_{-2k+1}\boxed{x_{-2k}}\cdots\boxed{x_1}\bullet\boxed{x_0}\boxed{x_3}\cdots\boxed{x_{2n+1}}\boxed{x_{2n}}x_{2n+3}\cdots \\
F_{02}^{(2)}(x) = & \cdots x_{-2k+3}\boxed{x_{-2k}}\cdots\boxed{x_3}\bullet\boxed{x_0}\boxed{x_5}\cdots\boxed{x_{2n+3}}\boxed{x_{2n}}x_{2n+5}\cdots \\
F_{02}^{(3)}(x) = & \cdots x_{-2k+5}\boxed{x_{-2k}}\cdots\boxed{x_5}\bullet\boxed{x_0}\boxed{x_7}\cdots\boxed{x_{2n+5}}\boxed{x_{2n}}x_{2n+7}\cdots \\
 & \quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots
\end{array}\right\} t$$

$$\left.\begin{array}{rl}
x = & \cdots x_{-2k-1}\overbrace{\boxed{x_{-2k}}\cdots\boxed{x_{-1}}\bullet\boxed{x_0}\boxed{x_1}\cdots\boxed{x_{2n-1}}\boxed{x_{2n}}}^{w}x_{2n+1}\cdots \\
F_{12}(x) = & \cdots x_{-2k+1}\boxed{x_{-2k+1}}\cdots\boxed{x_1}\bullet\boxed{x_1}\boxed{x_3}\cdots\boxed{x_{2n+1}}\boxed{x_{2n+1}}x_{2n+3}\cdots \\
F_{12}^{(2)}(x) = & \cdots x_{-2k+3}\boxed{x_{-2k+3}}\cdots\boxed{x_3}\bullet\boxed{x_3}\boxed{x_5}\cdots\boxed{x_{2n+3}}\boxed{x_{2n+3}}x_{2n+5}\cdots \\
F_{12}^{(3)}(x) = & \cdots x_{-2k+5}\boxed{x_{-2k+5}}\cdots\boxed{x_5}\bullet\boxed{x_5}\boxed{x_7}\cdots\boxed{x_{2n+5}}\boxed{x_{2n+5}}x_{2n+7}\cdots \\
 & \quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots
\end{array}\right\} t$$

One can easily check that, in all cases, $R(w,t) = 5^{w+(t-1)}$. Therefore, the entropies of periodic cellular automatas given by (11)- (14) are $\log 5$. □

5. Fractal Periodic Cellular Automata

In this section, we are going to provide some simulations on fractal structures of periodic as well as $(0-1)-$cellular automata.

Example 5.1. *Let us consider the following local rules*

$$f_{18}(1,1,1) = f_{18}(1,1,0) = f_{18}(1,0,1) = 0,$$

$$f_{18}(0,1,1) = f_{18}(0,1,0) = f_{18}(0,0,0) = 0,$$

$$f_{18}(1,0,0) = f_{18}(0,0,1) = 1,$$

and

$$f_{26}(1,1,1) = f_{26}(1,1,0) = f_{26}(1,0,1) = f_{26}(0,1,0) = f_{26}(0,0,0) = 0$$
$$f_{26}(0,1,1) = f_{26}(1,0,0) = f_{26}(0,0,1) = 1.$$

We define periodic cellular automata of period-2

$$[F_{eo}(x)]_i = \begin{cases} f_{18}(x_{i-1}x_ix_{i+1}) & i{=}even \\ f_{26}(x_{i-1}x_ix_{i+1}) & i{=}odd \end{cases},$$

$$[F_{oe}(x)]_i = \begin{cases} f_{18}(x_{i-1}x_ix_{i+1}) & i{=}odd \\ f_{26}(x_{i-1}x_ix_{i+1}) & i{=}even \end{cases}$$

and $(0-1)-$*cellular automata*

$$[F_{(0-1)}(x)]_i = \begin{cases} f_{18}(x_{i-1}x_ix_{i+1}) & i=1 \\ f_{26}(x_{i-1}x_ix_{i+1}) & i=0 \end{cases},$$

$$[F_{(1-0)}(x)]_i = \begin{cases} f_{18}(x_{i-1}x_ix_{i+1}) & i=0 \\ f_{26}(x_{i-1}x_ix_{i+1}) & i=1 \end{cases}$$

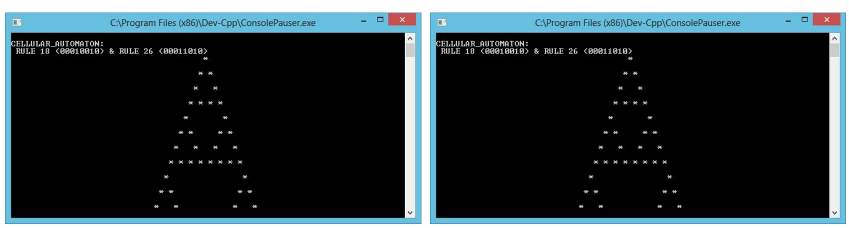

Figure 1: Simulation of periodic cellular automata F_{eo} and F_{oe}

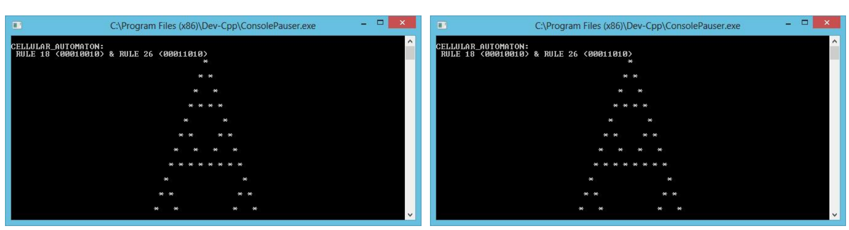

Figure 2: Simulation of periodic cellular automata $F_{(1-0)}$ and $F_{(0-1)}$

In both cases, we have fractal structures which are almost the same.

Example 5.2. *Let us consider the following local rules*

$$f_{129}(1,1,0) = f_{129}(1,0,1) = f_{129}(1,0,0) = 0,$$

$$f_{129}(0,1,1) = f_{129}(0,1,0) = f_{129}(0,0,1) = 0,$$

$$f_{129}(1,1,1) = f_{129}(0,0,0) = 1.$$

and

$$f_{161}(1,1,0) = f_{161}(1,0,0) = f_{161}(0,1,1) = f_{161}(0,0,1) = f_{161}(0,1,0) = 0,$$

$$f_{161}(1,1,1) = f_{161}(0,0,0) = f_{161}(1,0,1) = 1.$$

We define periodic cellular automata of period-2

$$[F_{eo}(x)]_i = \begin{cases} f_{129}(x_{i-1}x_ix_{i+1}) & i\text{=}even \\ f_{161}(x_{i-1}x_ix_{i+1}) & i\text{=}odd \end{cases},$$

$$[F_{oe}(x)]_i = \begin{cases} f_{129}(x_{i-1}x_ix_{i+1}) & i\text{=}odd \\ f_{161}(x_{i-1}x_ix_{i+1}) & i\text{=}even \end{cases},$$

and $(0-1)-$*cellular automata*

$$[F_{(0-1)}(x)]_i = \begin{cases} f_{129}(x_{i-1}x_ix_{i+1}) & i=1 \\ f_{161}(x_{i-1}x_ix_{i+1}) & i=0 \end{cases},$$

$$[F_{(1-0)}(x)]_i = \begin{cases} f_{129}(x_{i-1}x_ix_{i+1}) & i=0 \\ f_{161}(x_{i-1}x_ix_{i+1}) & i=1 \end{cases}.$$

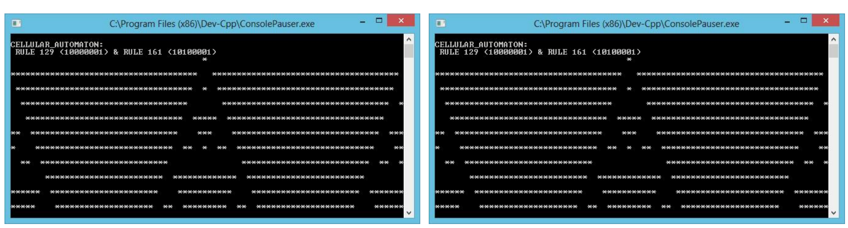

Figure 3: Simulation of periodic cellular automata F_{eo} and F_{oe}

In both cases, we have fractal structures. However they are different.

Example 5.3. *Let us consider the following local rules*

$$f_{126}(1,1,0) = f_{126}(1,0,1) = f_{126}(1,0,0) = 1,$$

$$f_{126}(0,1,1) = f_{126}(0,1,0) = f_{126}(0,0,1) = 1,$$

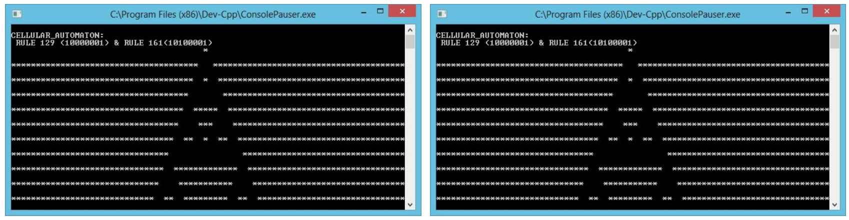

Figure 4: Simulation of periodic cellular automata $F_{(1-0)}$ and $F_{(0-1)}$

$$f_{126}(1,1,1) = f_{126}(0,0,0) = 0,$$

and

$$f_{182}(1,1,1) = f_{182}(1,0,1) = f_{182}(1,0,0) = f_{182}(0,1,0) = f_{182}(0,0,1) = 1,$$
$$f_{182}(1,1,0) = f_{182}(0,0,0) = f_{182}(0,1,1) = 0$$

We define periodic cellular automata of period-2

$$[F_{eo}(x)]_i = \begin{cases} f_{126}(x_{i-1}x_ix_{i+1}) & i{=}even \\ f_{182}(x_{i-1}x_ix_{i+1}) & i{=}odd \end{cases},$$

$$[F_{oe}(x)]_i = \begin{cases} f_{126}(x_{i-1}x_ix_{i+1}) & i{=}odd \\ f_{182}(x_{i-1}x_ix_{i+1}) & i{=}even \end{cases},$$

and $(0-1)-$*cellular automata*

$$[F_{(0-1)}(x)]_i = \begin{cases} f_{126}(x_{i-1}x_ix_{i+1}) & i=1 \\ f_{182}(x_{i-1}x_ix_{i+1}) & i=0 \end{cases},$$

$$[F_{(1-0)}(x)]_i = \begin{cases} f_{126}(x_{i-1}x_ix_{i+1}) & i=0 \\ f_{182}(x_{i-1}x_ix_{i+1}) & i=1 \end{cases}.$$

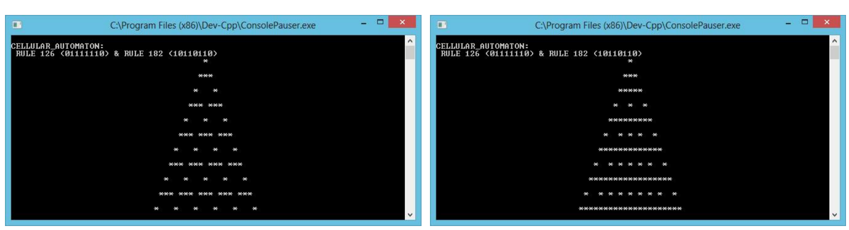

Figure 5: Simulation of periodic cellular automata F_{eo} and F_{oe}

It is clear that only $F_{(1-0)}$ *and* $F_{(0-1)}$ *have a fractal structure and the rest do not have it.*

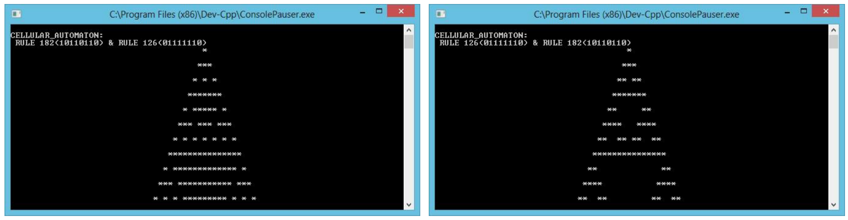

Figure 6: Simulation of periodic cellular automata $F_{(1-0)}$ and $F_{(0-1)}$

6. Conclusion

In this paper, we investigated periodic cellular automata of period-2 and $(0-1)-$cellular automata with two state spaces. It is recommended that the future studies will carry on the higher order periodic cellular automata with several state elements in an alphabet A. In this case, we can expect more interesting outcomes. Furthermore, we are aiming to compare the computational power of our models with the Turing Machine.

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